SAMPLE STUDY MATERIAL

INSTRUMENTATION ENGINEERING-IN

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Postal Correspondence Course

GATE & PSUs

COMMUNICATION THEORY

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CONTENT

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RANDOM SIGNALS & NOISE

Random Signals and Noise

Signals can be classified as deterministic and random. Deterministic signals can be completely specified as functions of time. Random signals are those whose value cannot be predicted in advance. Radio communication signals are random signals, which are invariably accompanied by noise. Noise like signals, also called random processes can be predicted about their future performance with certain probability of being correct.

Probability:

The concept of probability occurs naturally when we contemplate the possible outcomes of an experiment whose outcomes are not always the same. Thus the probability of occurrence of an event A, that is

$$P(A) = \frac{\text{number of possible favourable outcomes}}{\text{total number of possible equally likely outcomes}} \quad \text{... (i)}$$

Two possible outcomes of an experiment are defined as being mutually exclusive if the occurrence of one outcome precludes the occurrence of the other. Hence,

$$P(A_1 \text{ or } A_2) = P(A_1) + (A_2) \quad \text{... (ii)}$$

Where A_1 and A_2 are two mutually exclusive events.

The joint probability of related and independent events is given as:

$$P\left(\frac{B}{A}\right) = \frac{P(A, B)}{P(A)} \quad \text{... (iii)}$$

Where $P\left(\frac{B}{A}\right)$ is the conditional probability of outcome B, given A has occurred and P(A, B) is the probability of joint occurrence of A and B.

Random Variables:

The term random variable is used to signify a rule by which a real number is assigned to each possible outcome of an experiment.

A random experiment with sample space S. A random variable $X(\lambda)$ is a single valued real function that assigns a real number called the value of to each sample point λ of S.

The sample space S is termed as domain of random variable X and the collection of all numbers [values of] is termed as the range of random variable X.

The random variable X induces a probability measure on the real line

$$P(X = x) = P\{\lambda : X(\lambda) = x\}$$
$$P(X \le x) = P\{\lambda : X(\lambda) \le x\}$$
$$P(x_1 < x \le x_2) = P\{\lambda : x_1 < X(\lambda) \le x_2\}$$

If X can take on only a countable number of distinct values, then X is called a discrete random variable.

If X can assume any values within one are more intervals on the real line, then X is called a continuous random variable.

Cumulative Distribution Function (CDF):

The cumulative distribution function associated with a random variable is defined as the probability that the outcome of an experiment will be one of the outcome for which $X(A) \le x$, where x is any given number. If $F_x(x)$ is the cumulative distribution function then

$$F_x(x) = P(X \le x)$$

where $P(X \le x)$ is the probability of the outcome.

$F_x(x)$ has the following properties:

1. $0 \le F(x) \le 1$
2. $F(-\infty) = 0, \ F(\infty) = 1$
3. $F(x_1) \le F(x_2)$ if $x_1 < x_2$.

Probability Density Function (PDF):

The probability density function $f_x(x)$ is defined in terms of cumulative distribution function $F_x(x)$ as:

$$f_x(x) = \frac{d}{dx} F_x(x)$$

The properties of PDF are:

1. $f_x(x) \geq 0$, for all x

2. $$\int_{-\infty}^{\infty} f_x(x)\,dx = 1$$

3. $$F_x(x) = \int_{-\infty}^{x} f_x(x)\,dx$$

The relation between probability and probability density can be written as:

$$P'(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f(x)\,dx$$

Example 1:

Consider the probability $f(x) = ae^{-b|x|}$, where X is a random variable whose Allowable values range from $x = -\infty$ to $x = +\infty$.

Find:

a. The cumulative distribution function F (x).

b. The relationship between a and b

c. The probability that the outcome x lies between 1 and 2.

Solution:

a) The cumulative distribution function is :

$$F(x) = P(X \leq x)$$

$$= \int_{-\infty}^{x} f_x(x)\,dx$$

$$= \int_{-\infty}^{x} ae^{-b|x|}$$

$$= \begin{cases} \dfrac{a}{b}e^{bx}, & x \leq 0 \\ \dfrac{a}{b}(2-e^{-bx}) & x \geq 0 \end{cases}$$

$$\because \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\because \qquad \int_{-\infty}^{\infty} ae^{-b|x|}dx = \frac{2a}{b} = 1$

$\Rightarrow \qquad \frac{a}{b} = \frac{1}{2}$

c) The probability that x lies in the range between 1 and 2 is

$$P(1 \le x \le 2) = \frac{b}{2}\int_{1}^{2} e^{-b|x|}dx = \frac{1}{2}(e^{-b} - e^{-2b})$$

The average value of a random variable is given as:

$$\overline{X} = E(x) = m = \sum_i x_i P(x_i)$$

where $\overline{x}$ or E(x) or m represents the average, expected or mean value of random variable x.

The variance (σ^2) x is given as

$$\sigma^2 = E(x^2) - m^2$$

where σ is the standard deviation of x.

If the average value $m = 0$ then

$\sigma^2 = E(X)^2$

The covariance μ of two random variables x and y is defined as:

$$\mu = E\{(x - m_x)(y - m_y)\}$$

where m_i denotes the corresponding mean. For independent random variables $\mu = 0$.

The correlation coefficient ρ between the variables x and y, serves as a measure of the extent to which X and Y are dependent. It is given by

$$\rho = \frac{\mu}{\sigma_x \sigma_y} = \frac{E(xy)}{\sigma_x \sigma_y}$$

ρ falls within the range $-1 \le \rho \le 1$.

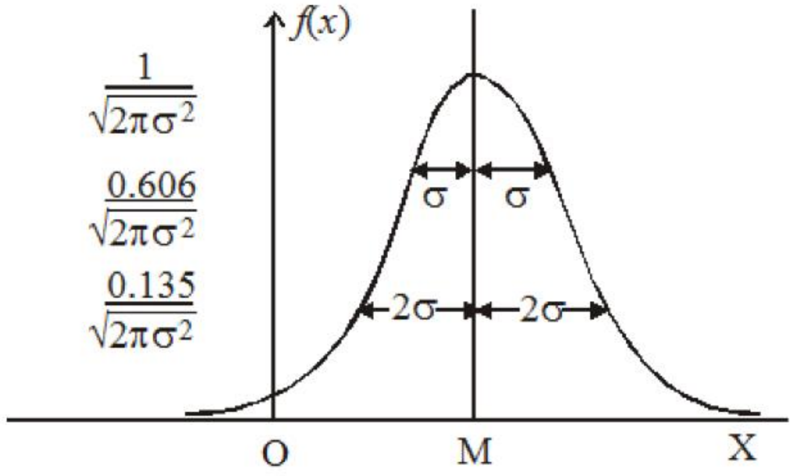
When $\rho = 0$, the random variables X and Y are said to be uncorrelated. When random variables are independent, they are uncorrelated but vice versa is not true.

Gaussian PDF:

The Gaussian PDF is of great importance in the analysis of random variables. It is defined as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-m)^2}{2\sigma^2}}$$

And is plotted in figure below, m and σ^2 are the average value and variance of f(x).



The Gaussian density Function

Thus,

$$\bar{x} = m = \int_{-\infty}^{\infty} x f(x)\, dx \text{ and}$$

$$E[(x-m)^2] = \int_{-\infty}^{\infty} (x-m)^2 f(x)\, dx = \sigma^2$$

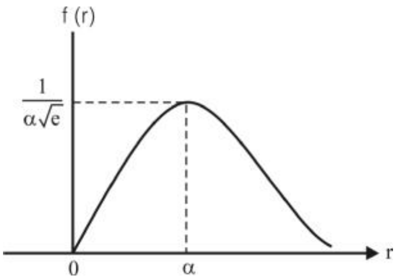
It can be deduced from figure that when $x - m = \pm\sigma$, $f(x)$ has fallen to 0.606 of its peak value. When $x = m \pm 2\sigma$, $f(x)$ falls to 0.135 of the peak value.

The Rayleigh PDF:

The Rayleigh PDF is defined by

$$f(r) = \begin{cases} \frac{r}{\alpha^2} e^{\frac{-r^2}{2\alpha^2}}, & 0 \le r \le \infty \\ 0, & r < 0 \end{cases}$$

A plot of $f(r)$ as a function of r is shown in figure below.



The Rayleigh PDF

The mean value is given by $\overline{R} = \sqrt{\frac{\pi}{2}}\alpha$, the mean squared value $\overline{R^2} = 2a^2$, and the variance $\sigma_r^2 = \left(2 - \frac{\pi}{2}\right)\alpha^2$.

Auto carrelation and Power Spectral Density (PSD):

A random process n(t), being neither periodic nor of finite energy has an autocorrelation function defined by

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{1}{2}} n(t)n(t+\tau)\,dt$$

Where τ is known as time lag parameter. For a random process the PSD G(f) is given as:

$$G(f) = F[R(\tau)] = \int_{-\infty}^{\infty} R(\tau)e^{jw\tau}dr$$

Thus, $G(f) \xleftrightarrow{F} R(\tau)$ they constitute a Fourier Transform pair.

Consider a deterministic waveform v (t), extending from $-\infty$ to ∞. Let a section of waveform V(t) be Selected such that $V_T(t) = v(t)$ for $\frac{-T}{2}$ to $\frac{T}{2}$ and zero otherwise. It $v_T(f)$ is the Fourier Transform of $v_T(t)$, $|v_T(f)|^2$ is the energy spectral density. Hence over the interval T the normalized power density is $\frac{|v_T(f)|^2}{T}$. As $T \to \infty$, $V_T(t) \to V(t)$

Thus, the physical significance of PSD G (f), is that

$$G(f) = \lim_{T\to\infty} \frac{1}{T}|V_T(f)|^2$$

Also, we can write the total energy of the signal V (t) is

$$E = \int_{-\infty}^{\infty} |G(f)|^2\,df$$

The is known as Rayleigh's Energy Theorem.

Properties of Autocorrelation function, R(²):

1. $R(\tau) = R(-\tau)$, i.e. it is an even function of time.
2. $R(0) = E[X^{2(t)}]$
3. $\max|R(\tau)| = R(0)$, i.e. $R(\tau)$ has maximum magnitude at zero time lag.

Properties of PSD, G(f):

1. G(f) is a real function of frequency f for a random process.
2. G(f) = G(-f), i.e., it is an even function of f.
3. G(f) ≥ 0 for all f.

Noise:

Noise is an unwanted signal of an unpredictable nature. It can either be man made ofrnatural. The internally generated noise due to circuit components is of two types. shot noise and thermal noise.

Shot Noise:

This type of noise is predominant in valves and transistors as current in these devices is due to flow of discrete electrons of diffusion of electrons and holes. The mean square value of shot noise current is:

$$I_{SN}^2 = 2eI_0\Delta f \text{ amp.}$$

Where, I_0- mean value of current (dc)

Δf-bandwidth of device in Hz.

Thermal Noise:

The noise in a conductor of resistor due to random motion of electrons is called thermal a noise. The mean sqare value of thermal noise voltage appearing across the terminals of the resistor R is

$$E_{TN}^2 = 4KTR\Delta f$$

Where,

T - the absolute temperature

K - Boltzmann's constant,

NoiseFigure:

Noise figure is defined as

$$= \frac{\text{Total available output power (due to device and source) per unit bandwidth}}{\text{Total available output noise power due to the source alone}}$$

= 10 $\log_{10}$ F, gives the Noise Factor

Terms of equlvalent noise temperature T, we can write.

$T_e = T_o(F-1)$

Where T_e = equivalent noise temperature of the amplifier or receiver whose Noise Figure is F.

T_o = 17º C = 290 K

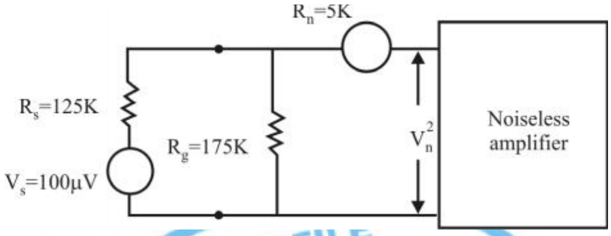
We can also write,

$$= \frac{(SNR)_{input}}{(SNR)_{output}}$$

Example 2:

A pentode having an equivalent noise resistance of 5 kΩ is selected for Equation as an amplifier with a signal source of emf 100 μV and an internal resistance 125 kΩ connected across the input. The grid leakage resistance is 175 kΩ. The width may be taken as 10KHz and the temperature as 27°C. Calculate the signal to --ratio at the output of trhe amplifier and express the value in dB. (Given 1.38x10^{23} J/K).

Solution:



Give input resistance $= \frac{R_s R_g}{R_s + R_g} = \frac{125 \times 175}{125 + 175} = 72.92 k\Omega$

$$100 \times \frac{175}{300} = 58.33\ \mu V$$

Voltage,

$= 4(Ra + R_{eff})$ KTB

$= 4(5 \times 10^3 + 72.92 \times 10^3) \times 1.38 \times 10^{-23} \times 300 \times 10^4$

$= 1290.36 \times 10^{-34}$ volts^2

$= 3.592\mu V$

$$\left(\frac{S}{N}\right)_{pentode} = \frac{V_{n4}^{\ 2}}{V_n^{\ 2}} = \left(\frac{58.33}{3.592}\right)^2 = 263.7$$

$$10\log_{10} 263.7 = 24.2dB$$

Example: A receiver connected to an antenna whose resistance is $50\,\Omega$ has an equivalent noise temperature of $30\,\Omega$. Calculate the receiver's noise figure in decibels and its equivalent noise temperature.

Sol. $F = 1 + \frac{R_{eq}}{R_o} = 1 + \frac{30}{50} = 1 + 0.6 = 1.6$

10 log 1.6 = 2.04 dB

$T_{eq} = T_o(F - 1) = 290(1.6 - 1) = 290 \times 0.6 = 174$ K

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INFORMATION THEORY

Information

The amount of information contained in an event is closely related to its uncertainty. For a discrete memoryless source (DMS), denoted by X, with symbols $\{x_1, x_2, \ldots x_m\}$, the information context of a symbol is given by

$$I(x_1) = \log\left(\frac{1}{P(x_i)}\right)$$

Unless mentioned take base as 2 since we are dealing with binary digits

$$I(x_i) = -\log_2 P(x_i) \quad \text{bits}$$

Average Information or Entropy:

Entropy of a source is given by

$$H(x) = -\sum_{i=1}^{m} P_i \log(P_i) = \sum_{i=1}^{m} P(x_i) I(x_i) \text{ bits / symbol}$$

When all the input digits have equal probability of occurrence we have maximum un-certainty and hence and hence maximum entropy.

For example say all the *m* events have equal probability of occurrence

$$P(x_1) = P(x_2) = P(x_3) = \ldots\ldots\ldots P(x_m) = \frac{1}{M}$$

$$\boxed{H_{max} = \log_2 m \text{ bits/ symbol}}$$

Example: A source is generating 3 possible symbols with probability $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$. Find the information associated with each of the symbol.

Solution: $P(x_1) = \frac{1}{4}$ $\qquad P(x_2) = \frac{1}{4}$

We know that $I(x_i) = +\log_2 \frac{1}{P(x_i)}$

Using above formula

$\mathrm{I}(x_1) = \log_2 4 = 2 \text{ bits}$ $\mathrm{I}(x_2) = \log_2(4) = 2 \text{ bits}$ $\mathrm{I}(x_3) = \log_2(2) = 1 \text{ bits}$

The probability of occurrence of x_3 is high hence information associated with it will be less

Information Rate:

If the time rate at which source x emits symbols is r (symbols/s), the information rate R of the source is given by

$$R = rH \text{ b/s}$$

Example: A source is generating 3 possible symbols with probabilities of $\frac{1}{4}, \frac{1}{4}, \frac{2}{4}$. Find entropy and information. Rate if the source is generating one symbol per millisecond.

Solution: We have

$$H = -\Sigma\, P(x_i)\log_2 P(x_i)$$

$$= -\left[\frac{1}{4}\times\log_2 4 + \frac{1}{4}\times\log_2 4 + \frac{1}{2}\times\log_2 2\right] = -\left[\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right]$$

$H = -\frac{3}{2}$ bits/symbols $\quad r = 1$ symbol/millisecond

Information rate (R) = H × r

$$= \frac{3}{2}\times\frac{1}{10^{-3}} \text{ bits/second}$$

$$R = \frac{3}{2} \text{ kbps}$$

Channel Capacity of a lossless channel:

$C = rC_s = r\log_2 m$, m-number of symbols in x.

Channel Capacity of Additive White Gaussian Noise Channel (AWGN):

The capacity of the AWGN channel is given by

$$C = B\log_2\left(1+\frac{S}{N}\right) \text{ b/s........Shannon-Hartley law}$$

where B-bandwidth

$\frac{S}{N}$ -SNR at the Channel output

Assuming Bandwidth to be infinite

Then

Shannon-Hartley law becomes.