SAMPLE STUDY MATERIAL

Postal Correspondence Course

GATE, IES & PSUs

Civil Engineering

Theory of Structures

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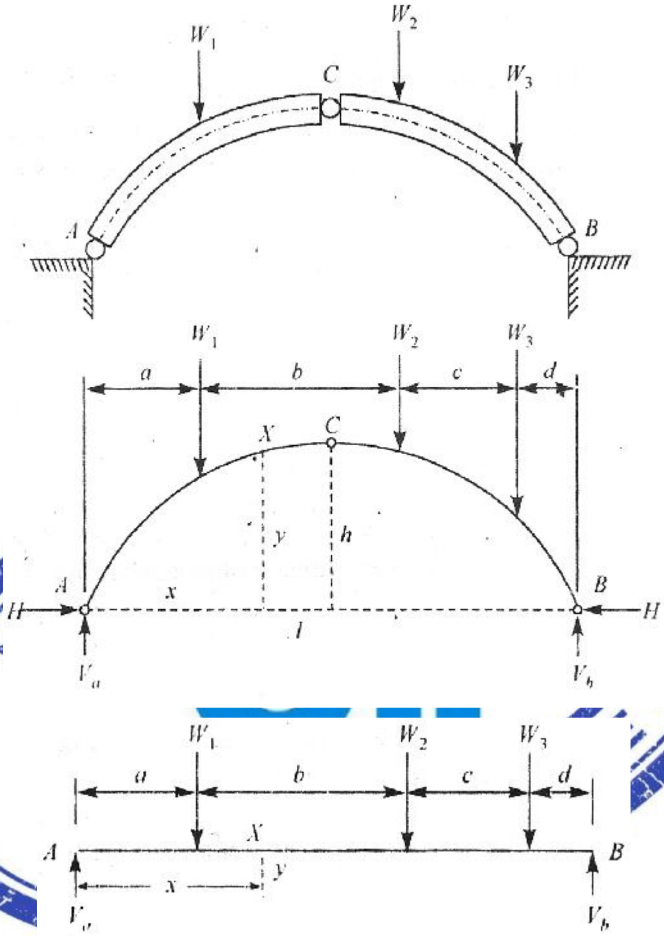
CONTENT

CHAPTER-1

ARCHES

Three Hinged Arches:

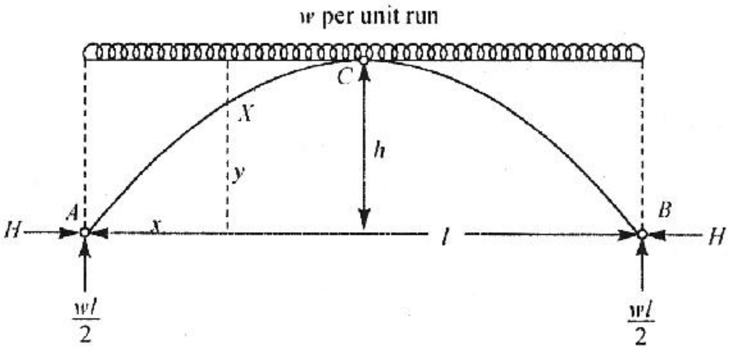


- As shown above, the arch is hinged at ‘A’, ‘B’ and ‘C’.
- The horizontal distance between the lower hinges ‘A’ and ‘B’ is called **‘span’ of** the arch.
- The hinges ‘A’ and ‘B’ may or may not be at the same level.
- When two lower hinges (A and B) are at the same level the height of the crown (highest point of the arch) above the level of the lower hinges is called **rise** of the arch.
- The difference between the beam and the arch is that in the case of the arch a horizontal thrust **(H)** is induced at each support which provides a hogging moment ‘Hy’ at any section.
- If moment ‘Hy’ is called **H moment at the section X-X.**

  Hence, [Actual bending moment at section X = Beam moment at section X-X –H moment at X-X]
- The sectional requirement for an arch is less than that of a beam of the same span and carrying the same load system.

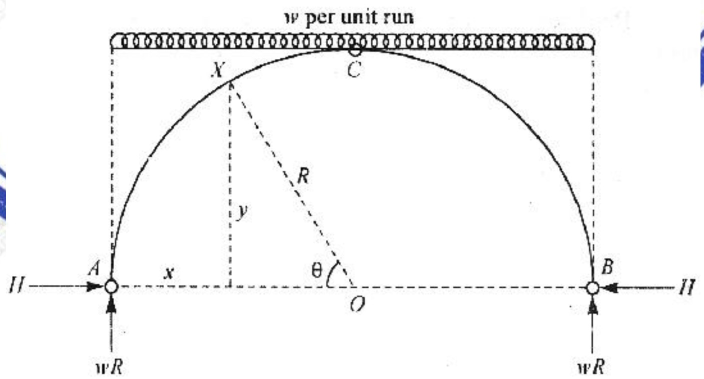
SOME ILLUSTRATIONS:

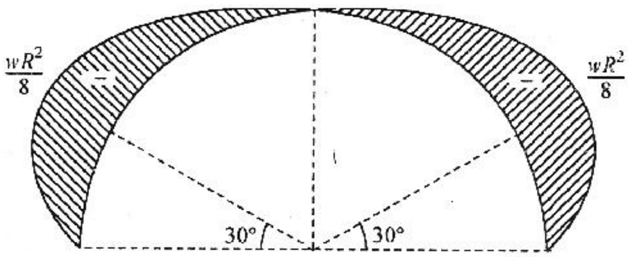
Case I: A three hinged arch of span *l* and rise *h* carrying a uniformly distributed load of *w* per unit run over the span.



- Horizontal thrust at each support, $H = \frac{wl^2}{8h}$
- Bending moment at any section of the arch is zero, $\boxed{M = 0}$
- The equation of the arch with end 'A' as origin is given by: $\boxed{y = \frac{4h}{l^2}(l - x) - x}$

Case II: A three hinged semicircular arch of the radius 'R' carrying a uniformly distributed load of *w* per unit run over the whole span.

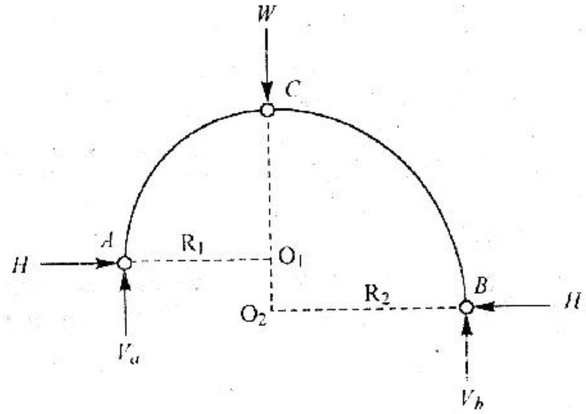




- Horizontal thrust at each end, $H = \frac{wR}{2}$.
- Bending moment at any section X-X making an angle 'θ' with the horizontal is given by: $\boxed{M_x = -\frac{wR^2}{2}[\sin^{-} - \sin^{2-}]}$ (hogging moment)

- Maximum bending moment, $M_{max.} = \frac{wR^2}{8}$ At an angle $\bar{\ } = 30°$ i.e. Distance of the point of maximum bending moment from the crown $= R\cos 30° = \frac{R\sqrt{3}}{2}$.

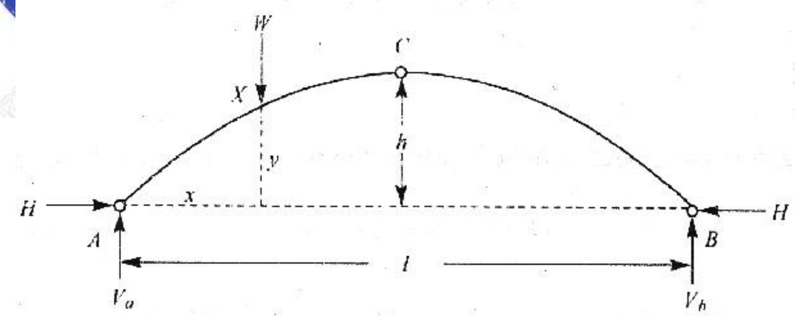
Case III: A three hinged arch consisting of two quadrant parts AC and CB of radii R_1 and R_2 carrying a concentrated load 'W' on the crown.



- Horizontal thrust at each end, $H = \frac{W}{2}$
- Reactions at each end equal to the horizontal thrust

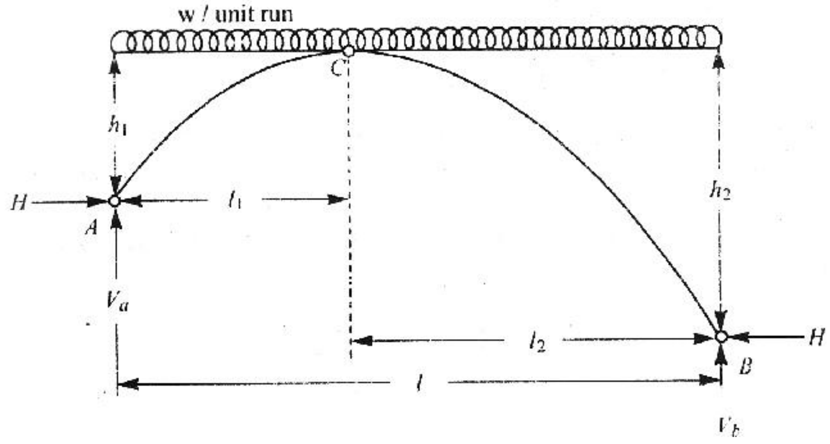
$$V_a = V_b = H = \frac{W}{2}$$

Case IV: A symmetrical three hinged parabolic arch of span '*l*' and rise '*h*' carrying a point load '*W*' which may be placed anywhere on the span.



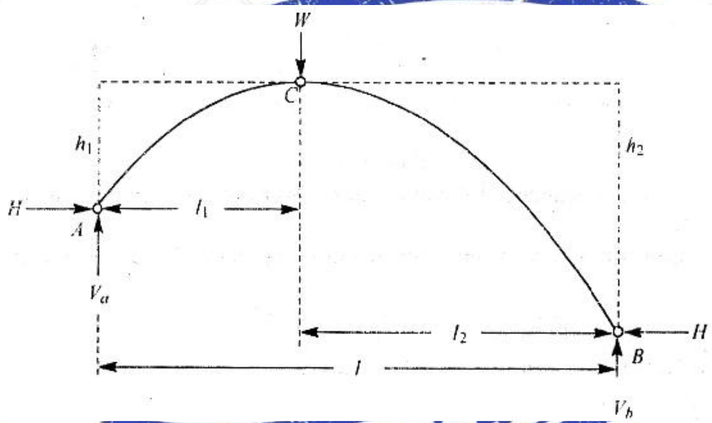
- Horizontal thrust at each end, $H = \frac{Wx}{2h}$
- Maximum Bending momentoccurs under the load.
- For the condition of absolute maximum bending moment, $x = \frac{l}{2\sqrt{3}}$ on either side of the crown

Case V: A three hinged parabolic arch of span '*l*' having its abutments at depth h_1 and h_2 below the crowns carrying a uniformly distributed load of w per unit run over the whole span.



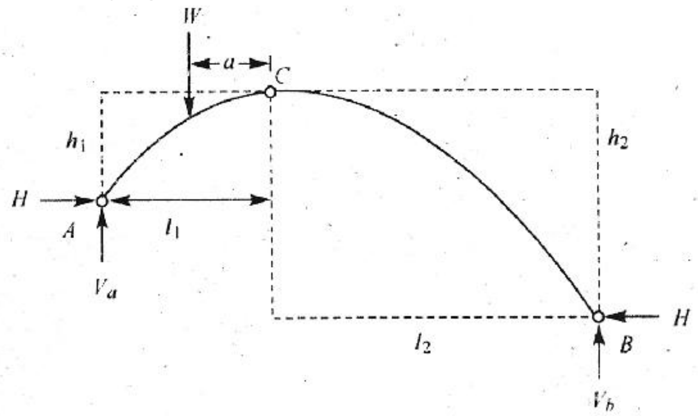
➢ Horizontal thrust at each end, $H = \dfrac{wl^2}{2\left(\sqrt{h_1}+\sqrt{h_2}\right)^2}$

Case VI: A three hinged parabolic arch of span '*l*' having its abutments 'A' and 'B' at depths h_1 and h_2 below the crown carrying a concentrated load '*W*' at the crown.



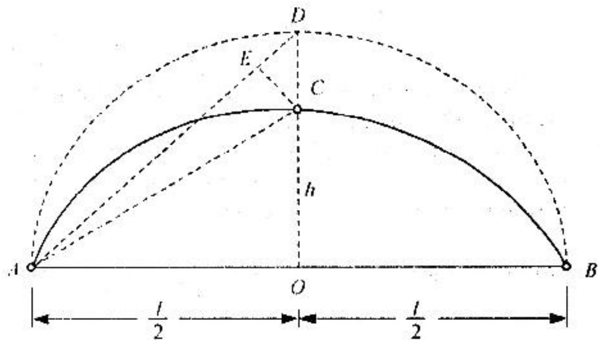
➢ Horizontal thrust at each end, $H = \dfrac{Wl}{\left(\sqrt{h_1}+\sqrt{h_2}\right)^2}$

Case VII: A three hinged parabolic arch of span '*l*' having its abutments 'A' and 'B' at depths h_1 and h_2 below the crown carrying a concentrated load '*W*' at a distance '*a*' from crown.



➢ Horizontal thrust at each end. $H = \dfrac{Wl_2(l_1-a)}{h_1 l_2 + h_2 l_1}$

Temperature effect on three-hinged arch:



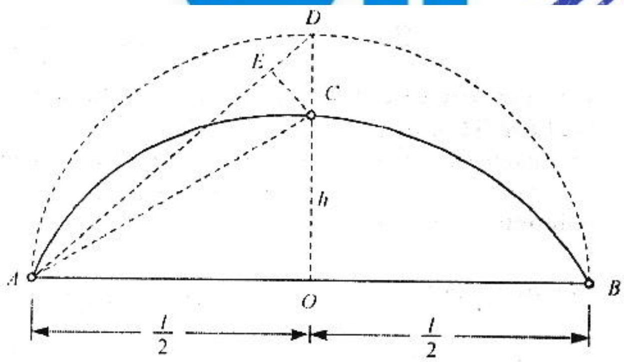
- Rise in temperature increases the length of the arc. Since the ends A and B do not move and since the hinge *C* is not connected to any permanent object, the crown hinge will rise from C to D. 'AD' represents the new position of AC.
- Increase in the rise of arch $= CD =$ ¢ $= \frac{l^2 + 4h^2}{4h} \propto T$,

  Where, T = change in temp. (ºC)

  $\propto$ = Co-efficient of linear expansion
- Due to temperature change, stresses are not produced in the arch, but the horizontal thrust changes.

$\frac{dH}{H} = -\frac{dh}{h}$ i.e. Horizontal thrust decreases due to rise in temperature.

Two hinged arches:



- Two hinged arch is an indeterminate arch.
- Strain energy stored in the whole arch, $U_i = \int (M - Hy)^2 \frac{ds}{2EI}$ where, *M* = B.M. of beam at section X-X
- By First theorem of Castigliano, the horizontal thrust can be obtained using

$$\frac{\partial U_i}{\partial H} = 0$$

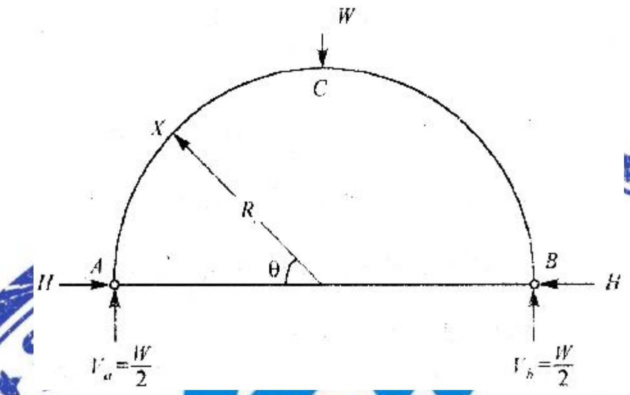
Horizontal thrust, $H = \dfrac{\int \frac{Myds}{EI}}{\int \frac{y^2 ds}{EI}}$

If flexural rigidity of arch is uniform, $H = \dfrac{\int Myds}{\int y^2 ds}$

- For a parabolic arches, it can also be given as: $H = \dfrac{\int Mydx}{\int y^2 dx}$

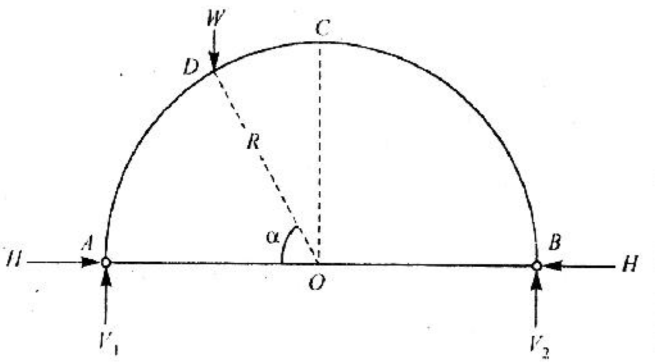
SOME ILLUSTRATIONS:

Case I: A two-hinged semicircular arch of radius 'R' carrying a concentrated load 'W' at the crown. Flexural rigidity (EI) is constant.



- Horizontal thrust, $H = \dfrac{W}{®}$
- Horizontal thrust is independent of Radius of the arch.
- Vertical deflection of the crown, $¢ = \dfrac{WR^3}{8® EI}(3®^2 - 8® - 4)$

Case II: A two-hinged semicircular arc of radius 'R' carrying a load W at a section the radius vector corresponding to which makes an angle $\propto$ with the horizontal. EI is constant.

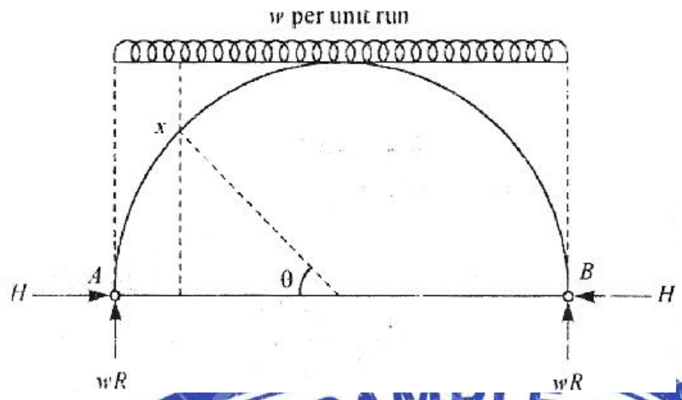


- Horizontal thrust, $H = \dfrac{W}{®}\sin^2 \sim$

- If there are loads $W_1, W_2, W_3.....$ at an angle $\propto_1, \propto_2, \propto_3$ .... (Less than or equal to 90°); Horizontal thrust, H

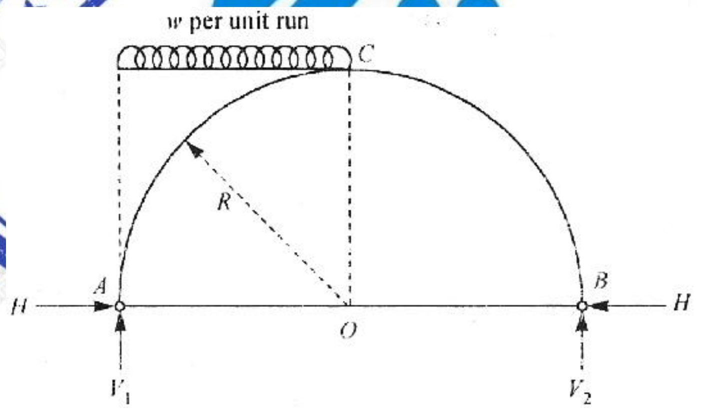
$$H = \sum \frac{W_i}{\text{®}} \sin^2 \propto_i$$

Case III: A two hinged semicircular arch of radius R carrying a uniformly distributed load w per unit run over the whole span. EI = constant.



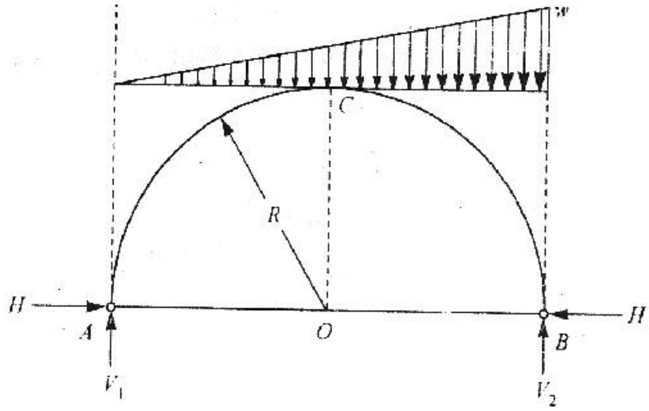
- Horizontal thrust, $H = \frac{4}{3} \cdot \frac{wR}{\text{®}}$

Case IV: A two hinged semicircular arch carrying a uniformly distributed load of w per unit run over the left half of its span. EI = constant.



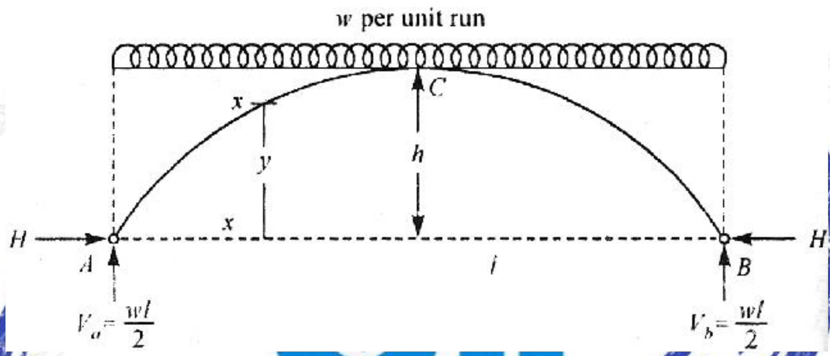
- Horizontal thrust, $H = \frac{2}{3} \cdot \frac{WR}{\text{®}}$.

Case V: A two-hinged semicircular arch of radius R carrying a distributed load uniformly varying from zero at the left end to w per unit run at the right end. EI = constant



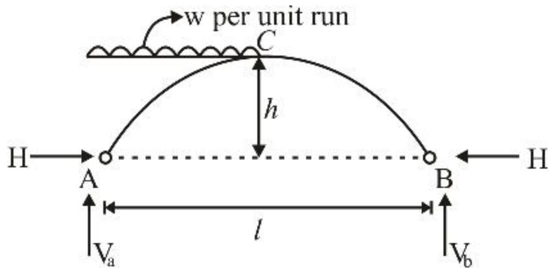
- Horizontal thrust, $H = \frac{4}{3} . \frac{wR}{®}$.

Case VI: A two-hinged parabolic arch of span *l* and rise *h* carrying a uniformly distributed load *w* per unit run over the whole span .*EI* = constant



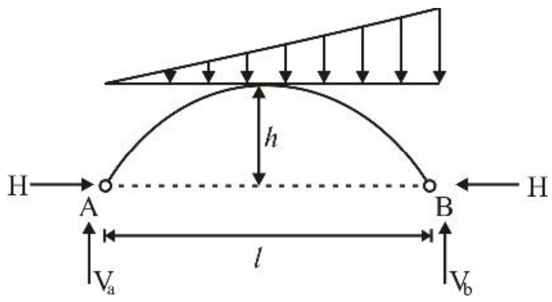
- Horizontal thrust, $H = \frac{wl^2}{8h}$,

Case VII: A two-hinged parabolic arch carrying a u.d.l. of *w* per unit run on its left half of the span.
EI = constant



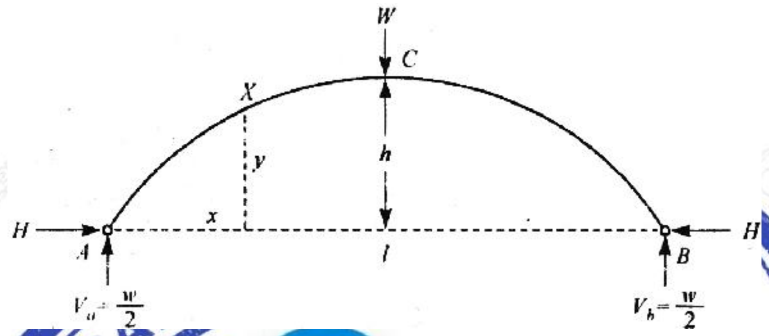
- Horizontal thrust, $H = \frac{wl^2}{16h}$.

Case VIII: A two hinged parabolic arch of span '*l*' and rise '*h*' carrying a load varying uniformly from zero at the left end to *w* per unit run at right end. *EI* = constant.



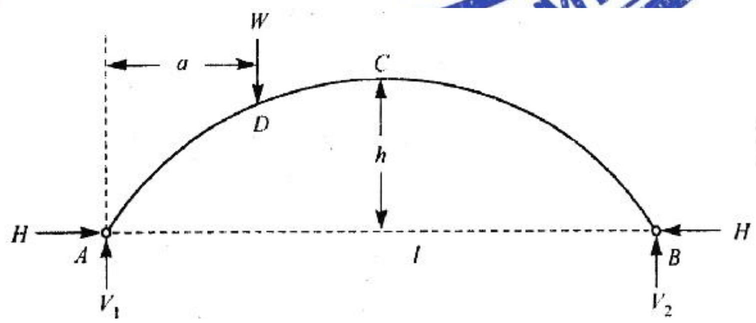
- Horizontal thrust, $H = \frac{wl^2}{16h}$

Case IX: A two hinged parabolic arch of span *l* and rise *h* carrying a concentrated load *W* at the crown. EI = constant.



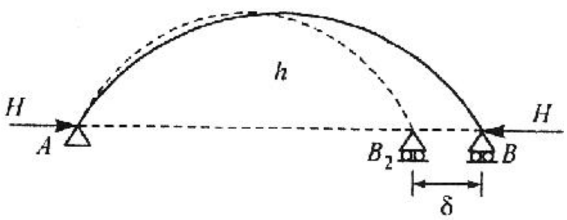
- Horizontal thrust, $H = \frac{25}{128}.\frac{Wl}{h}$

Case X: A two-hinged parabolic arch of span *l* and rise *h* carrying a concentrated load *W* at a distance '*a*' from the left end. EI = constant



- Horizontal thrust, $H = \frac{5}{8}.\frac{W}{hl^3}a(l-a)(l^2+la-a^2)$

Temperature effect on two-hinged arches:



- Due to increase in temperature, the length of member tends to increase. Since both ends are hinged and displacement cannot take place, inward horizontal thrust will be developed at each support.
- Horizontal thrust, $H = \dfrac{EI \propto Tl}{\int y^2 ds}$

  Where, $\propto$ = Coefficient of linear expansion

  T = change in temperature (ºC).

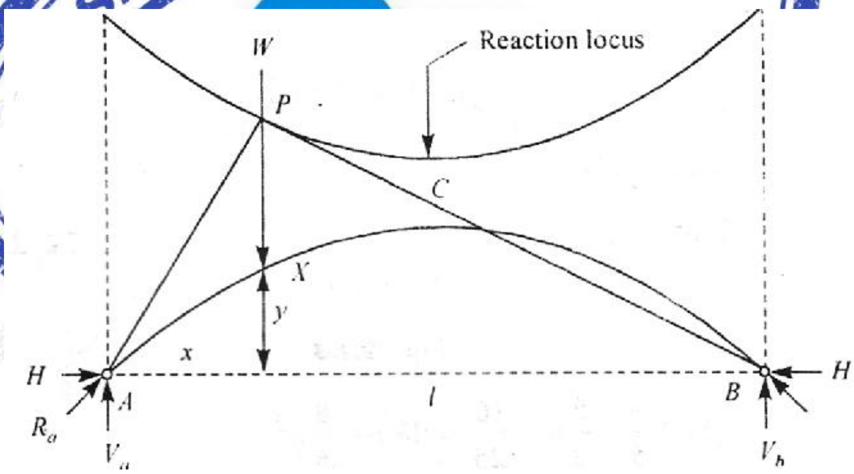
  For semi-circular two-hinged arch; $H = \dfrac{4EI \propto T}{® R^2}$

  For parabolic two-hinged arch, $H = \dfrac{EI_0 \propto Tl}{\int y^2 dx} = \dfrac{15}{8} \cdot \dfrac{EI_0 \propto T}{h^2}$,

- Where $I = I_0 \sec ¯$ At $¯ = 0,\ I = I_0 = M.O.I.$ at the crown

Reaction locus for a two-hinged arch:

- The reaction locus for a two-hinged arch is the locus of the point of intersection of the two resultant reactions at the supports as a point load moves on the span of the arch.

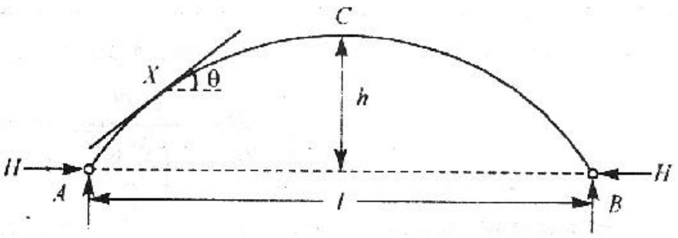


Note:

- Reaction locus for a two-hinged semicircular arch is a straight line parallel to the line joining abutments and at a height of $\dfrac{® R}{2}$ above it.
- Reaction locus for a two-hinged parabolic arch is a curve whose equation is given by

  $y = \dfrac{1.6hl^2}{l^2 + lx - x^2}$ where '*x*' is distance from end 'A'

Effect of rib shortening:



- At any section of the arch, a bending moment a shear force and a normal thrust act. The normal thrust causes a shortening of the actual length of the arch.
- For this condition Horizontal thrust, (Two-hinged arch)

$$H = \frac{\int \frac{Myds}{EI} - \int \frac{V\sin\alpha .\cos\alpha .ds}{AE}}{\int \frac{y^2 ds}{EI} + \int \frac{\cos^2 \alpha .ds}{AE}}$$

- Combined effect of rib shortening and temperature rise, horizontal thrust,

$$H = \frac{\alpha Tl}{\int \frac{y^2 ds}{EI} + \frac{1}{A_m E}}$$ where, A_m = Mean value of $\frac{A}{\cos\alpha}$.

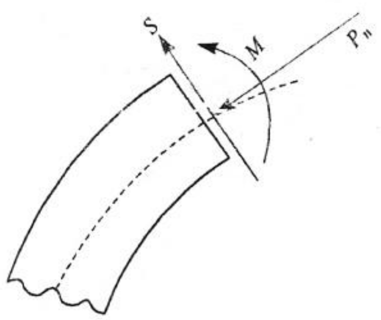
Two-Hinged arch with laterally yielding supports:

- If one support yield laterally by 'δ' with respect to other support. horizontal thrust,

$$H = \frac{\int \frac{Myds}{EI}}{\int \frac{y^2 dx}{EI} + K}$$ where $\delta = KH$

- Hence, combined effect of temperature rise, rib shortening and laterally yielding of support, horizontal thrust, $H = \dfrac{\int \frac{Mydx}{EI_0} + \alpha Tl}{\int \frac{y^2 dx}{EI_0} + \frac{1}{A_m E} + K}$

Normal Thrust and Radial Shear:



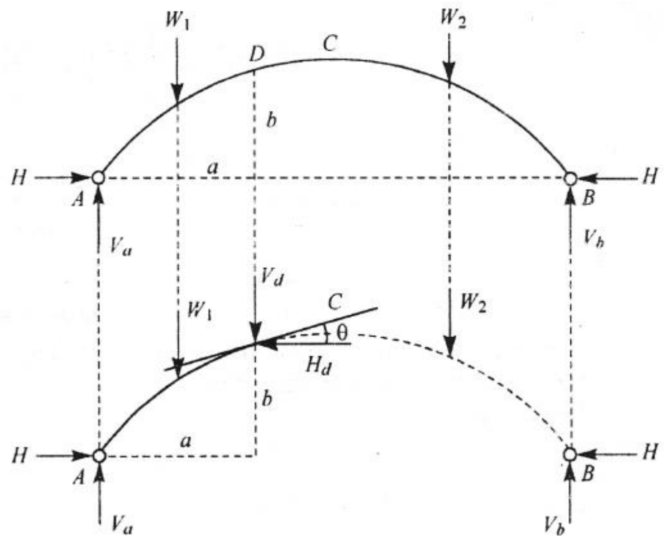


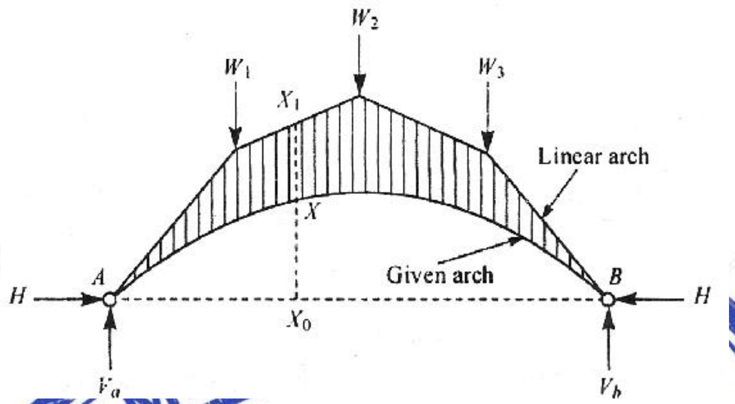
Figure: Arch section subjected to normal thrust P_n radial shear S bending Moment M

Figure

- Component of reacting forces at D along the tangent is called **Normal thrust** at D. $\boxed{\text{Normal thrust at } D = P_n = H_d \cos\bar{} + V_d \sin\bar{}}$
- Component of reacting forces at D perpendicular to the tangent is called **Radial shear** or **simply shear** at D.

$\boxed{\text{Radial Shear at } D = S = H_d \sin\bar{} - V_d \cos\bar{}}$

The lines arch OR The theoretical arch OR The line of thrust:

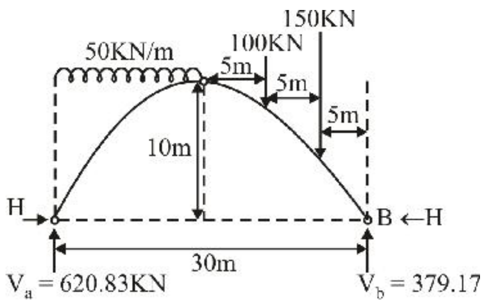


- The different members of the linear arch are subjected to axial compressive forces only. The joints of this linear arch are in equilibrium.
- The shape of the linear arch follows the shape of the free bending moment diagram for a beam of the same span and subjected to same loading.
- Bending moment at any section of an arch is proportional to the ordinate or the intercept between the given arch and the linear arch. This principle is called **Eddy's theorem.**
- The actual Bending moment at the section 'X' is proportional to the ordinate $X_1 X_0$.

Practice: Questions & Solutions

1. A three hinged arch has a span of 30 meters and a rise of 10 m. The arch carries a uniformly distributed load of 50 KN per meter on the left half of the span. It also carries two concentrated loads of 150 KN. and 100 KN at 5m and 10m from the right end. Calculate the horizontal thrust at each support.

Solution:



H → Horizontal thrust

V_a→Vertical reaction at 'A'

V_b→ Vertical reaction at 'B'

Total vertical load, V = 50 × 15 + 100 + 150 = 1000 KN.

Taking moment about the end 'A' ∴$\Sigma M_A = 0$

$$\therefore \quad V_b \times 30 = (50\times15)\times\frac{15}{2} + 100\times20 + 150\times25$$

$$\Rightarrow \quad V_b \times 30 = 11375 \quad \therefore \quad \boxed{V_b = 379.17kN.}$$

$$\because \quad V_a + V_b = V$$

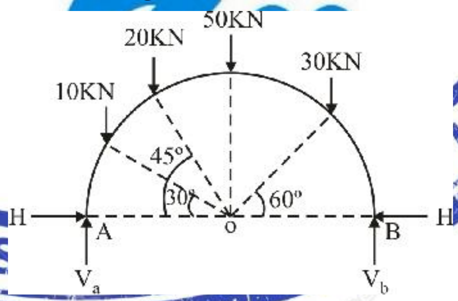
$$\boxed{V_a = 1000 - 379.17 = 620.83kN}$$

Taking moment about 'C': $\sum M_c = 0$

$$620.83\times15 - H\times10 - 50\times15\times\frac{15}{2} = 0$$

$$\Rightarrow \quad 3687.45 = 10H \quad \boxed{H = 368.745kN.}$$

2. Find the horizontal thrust the two-hinged semicircular arch loaded as shown in figure.



Solution: Horizontal thrust, $H = \sum \frac{\mu}{®}\sin^2 \sim$

$$= \frac{10}{®}\sin^2 30° + \frac{20}{®}\sin^2 45° + \frac{50}{®}\sin^2 90° + \frac{30}{®}\sin^2 60°$$

$$= \frac{85}{®} = 27.06\,KN.$$

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